Preliminary Solutions for exam February 21, 2008
10/08

1. a) $P(W)=P(W\mid B_1)P(B_1)+P(W\mid B_2)P(B_2)+P(W\mid B_3)P(B_3)+P(W\mid B_4)P(B_4)$

$$=1\cdot\frac{1}{2}+0\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{8}+\frac{3}{4}\cdot\frac{1}{8}=\frac{21}{32}$$

b) $P(B_1\mid W)=\dfrac{P(B_1\cap W)}{P(W)}=\dfrac{P(W\mid B_1)P(B_1)}{P(W)}=\dfrac{1\cdot\frac{1}{2}}{\frac{21}{32}}=\dfrac{16}{21}.$

2. a) $f_1(y_1)=\int\limits_{-\infty}^{\infty} f(y_1,y_2)dy_2=\int\limits_{y_1}^{1}6(1-y_2)dy_2=6(y_2-\frac{y_2^2}{2})\Big]_{y_1}^{1}=3(1-y_1)^2 \quad \text{for } 0\le y_1\le 1,$

$$f_2(y_2)=\int\limits_{-\infty}^{\infty} f(y_1,y_2)dy_1=\int\limits_{0}^{y_2}6(1-y_2)dy_1=\left[6y_1-6y_1y_2\right]_0^{y_2}=6y_2(1-y_2) \quad \text{for } 0\le y_2\le 1,$$

b) $\mathrm{E}(\mathrm{Y}_1)=\frac{1}{4},\ E(Y_2)=\frac{1}{2},$

$$E(Y_1Y_2)=\int\limits_0^1\int\limits_0^{y_2}6y_1y_2(1-y_2)dy_1dy_2=\int\limits_0^1 3(y_2^3-y_2^4)dy_2=\frac{3}{4}-\frac{3}{5}=\frac{3}{20},$$

$$COV(Y_1,Y_2)=\frac{3}{20}-\frac{1}{8}=\frac{1}{40},$$

$\therefore$ Since $COV(Y_1,Y_2)\neq 0$, Y_1 and Y_2 are not independent.

c) $\mathrm{E}(\mathrm{Y}_1)=\int\limits_0^1 y_1\cdot f(y_1)dy_1=\int\limits_0^1 3y_1(1-y_1)^2dy_1=\int\limits_0^1(3y_1-6y_1^2+3y_1^3)=\frac{1}{4},$

$$E(Y_2)=\int\limits_0^1 y_2\cdot f(y_2)dy_2=\int\limits_0^1 6\cdot y_2^2\cdot(1-y_2)dy_2=\left[2y_2^3-\frac{3}{2}y_2^4\right]_0^1=\frac{1}{2},$$

$$E(Y_1-3Y_2)=E(\left(Y_1\right)-3E(Y_2)=\frac{1}{4}-3\cdot\frac{1}{2}=-\frac{5}{4}.$$

3. a) $\hat{p}_1$ and $\hat{p}_2$ are unbiased estimators

$E(\hat{p}_1) = p, \ E(\hat{p}_2) = p,$

$$E(p_1^*) = \frac{1}{2}(p + p) = p$$

$$E(p_2^*) = E(\frac{n_1\hat{p}_1 + n_2\hat{p}_2}{n_1 + n_2}) = \frac{n_1}{n_1 + n_2}E(\hat{p}_1) + \frac{n_2}{n_1 + n_2}E(\hat{p}_2)$$

$$= p(\frac{n_1}{n_1 + n_2} + \frac{n_2}{n_1 + n_2}) = p,$$

$\therefore$ Both estimators are unbiased.

b) $Var(p_1^*) = Var(\frac{\hat{p}_1 + \hat{p}_2}{2}) = \frac{1}{4}\{Var(\hat{p}_1) + Var(\hat{p}_2)\}$

$$= \frac{1}{4}\left[\frac{p(1-p)}{n_1} + \frac{p(1-p)}{n_2}\right] = \frac{p(1-p)(n_1 + n_2)}{4n_1n_2},$$

$$Var(p_2^*) = Var(\frac{n_1\hat{p}_1 + n_2\hat{p}_2}{n_1 + n_2}) = (\frac{n_1}{n_1 + n_2})^2 Var(\hat{p}_1) + (\frac{n_2}{n_1 + n_2})^2 Var(\hat{p}_2)$$

$$= (\frac{n_1}{n_1 + n_2})^2(\frac{p(1-p)}{n_1}) + (\frac{n_2}{n_1 + n_2})^2(\frac{p(1-p)}{n_2}) = \frac{p(1-p)}{(n_1 + n_2)},$$

$$Var(p_1^*) - Var(p_2^*) = \frac{p(1-p)(n_1 + n_2)}{4n_1n_2} - \frac{p(1-p)}{(n_1 + n_2)}$$

$$= p(1-P)\left[\frac{(n_1 + n_2)^2 - 4n_1n_2}{4n_1n_2(n_1 + n_2)}\right] = \frac{p(1-P)}{4n_1n_2(n_1 + n_2)}(n_1^2 + n_2^2 + 2n_1n_2 - 4n_1n_2)^2$$

$$= \frac{p(1-P)}{4n_1n_2(n_1 + n_2)}(n_1 - n_2)^2 > 0$$

$\therefore \ Var(p_2^*) < Var(p_1^*), \quad p_2^*$ is more efficient.

Alternatively,

$$eff(p_1^*, p_2^*) = \frac{Var(p_2^*)}{Var(p_1^*)} = \frac{\frac{p(1-p)}{(n_1 + n_2)}}{\frac{p(1-p)(n_1 + n_2)}{4n_1n_2}} = \frac{4n_1n_2}{(n_1 + n_2)^2}$$

$(n_1 + n_2)^2 - 4n_1n_2 = (n_1 - n_2)^2 > 0,$

$\therefore \ (n_1 + n_2)^2 > 4n_1n_2, \qquad \therefore \ eff(p_1^*, p_2^*) < 1,$

$\therefore \ Var(p_2^*) < Var(p_1^*), \quad p_2^*$ is more efficient.

4. $Y \sim \exp(\beta), \qquad f(y) = \frac{1}{\beta} e^{-y/\beta}, \quad \beta > 0,$

$$L(y) = \prod_{i=1}^{n} f(y_i) = \prod_{i=1}^{n} \frac{1}{\beta} e^{-y_i/\beta} = \frac{1}{\beta^n} e^{-\Sigma y_i/\beta},$$

$$LnL(y) = -nLn\beta - \frac{1}{\beta}\Sigma y_i,$$

$$\frac{dLnL(y)}{dy} = -\frac{n}{\beta} + \frac{\Sigma y_i}{\beta^2} = 0, \quad \Rightarrow \quad \hat{\beta} = \frac{1}{n}\Sigma y_i = \bar{y}.$$

5. a) $H_0: \mu = 800 \qquad \alpha = 0.05$

$H_1: \mu < 800$

$n = 5$, a t-test should be applied.

The sample mean and variance are

$$\bar{y} = \frac{1}{5}\Sigma_{i=1}^{5} y_i = \frac{1}{5}(785 + 805 + ... + 802) = 795.$$

$$s^2 = \frac{1}{n-1}\Sigma_{i=1}^{5}(y_i - \bar{y})^2 = 69.5.$$

test statistic: $t = \dfrac{\bar{y} - \mu_0}{s/\sqrt{n}}$

RR: Reject H_0 if $t_{obs} < t_{(0.05, 5-1)} = -2.132$

$$t_{obs} = \frac{795 - 800}{\sqrt{69.5/5}} = -1.341 > -2.132,$$

Non significant result. We do not have enough evidence to reject H_0.

b) Assumptions:

- random sample;
- independent observations;
- normally distributed population.

c) $p - value = P(t \le -1.341)$

from t-table: $t_{0.10,4} = 1.533$

since $-1.341 > 1.533$,

$p - value > 0.10.$