Statistics B3: Time Series Analysis
Solutions to Time Series Analysis exam
2016-04-29

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Task 1

A)

A stochastic process is a sequence of random variables.

B)

In order to be *covariance stationary*, a stochastic process need to fulfill the following

- $E[Y_t] = \mu, \, \forall \, t$, i.e. the mean is constant over time.
- $V(Y_t) = \gamma_0 < \infty \, \forall \, t$, i.e. the variance is finite and constant over time.
- $Cov(Y_{t+j}, Y_t) = Cov(Y_j, Y_0) \, \forall \, t, j$, i.e. the covariance is independent of *t* and only a function of the lag length *j*.

The third condition actually implies the second condition.

C)

A *trend-stationary* process is (covariance) stationary when removing the trend. The trend should be a function of time only. We can specify such a process by

$$Y_t = \alpha_0 + \alpha_1 t + e_t,$$

where $e_t \sim iid\,(0, \sigma^2)$. By removing the trend we get

$$Y_t - \alpha_1 t = \alpha_0 + e_t$$

which is stationary.

D)

A *difference-stationary* process is a process that is (covariance) stationary when taking difference. An example of this is a standard random walk, i.e.

$$Y_t = Y_{t-1} + e_t,$$

where $e_t \sim iid\,(0, \sigma^2)$. Taking first difference we get

$$Y_t - Y_{t-1} = e_t,$$

which is stationary.

E)

- **Identification**: Identify the DGP, or find a DGP that is suitable for modelling the data, this can be done by looking at correlograms.
- **Estimation**: Estimate the parameters of the model, this can be done by OLS or maximum likelihood estimation.
- **Diagnostics**: Investigate to what extent the model captures the systematic variation in the data. The residuals should be white noise, i.e. uncorrelated. This can be done by looking at correlograms and performing statistical tests on the estimated parameters and residuals.
- **Forecasting**: Use the model to forecast conditional mean of the process. Forecasts can be done based on conditional means.

F)

1.

The fourth autocorrelation is the correlation between the stochastic process at a given time point and the same process four time points back.

2.

The fourth partial autocorrelation is the correlation between the stochastic process at a given time point and the same process four time points back, when taking the autocorrelations 1-3 into account.

Task 2

A)

1)

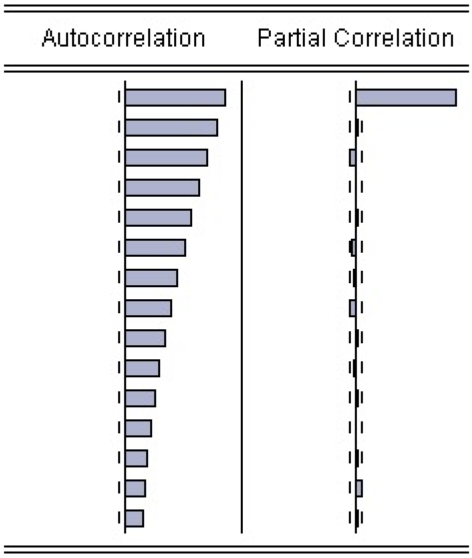
$$SARMA(1,0) \times (0,0)_0$$

2)

$y_t = \phi y_{t-1} + e_t$

3)

For $\phi = 0.9$:



B)

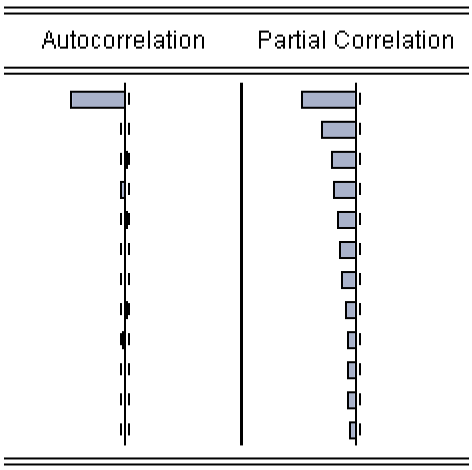
1)

$$SARMA(0,1)\times(0,0)_0$$

2)

$$y_t = e_t - \theta_1 e_{t-1}$$

3)



C)

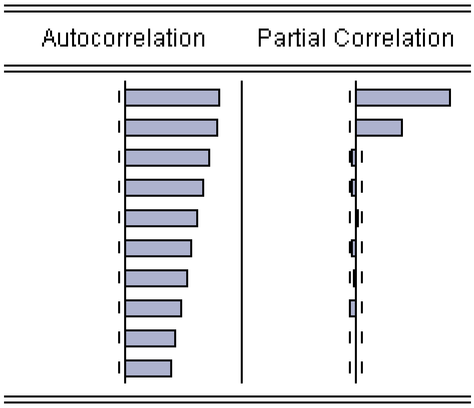
1)

$$SARMA(2,0)\times(0,0)_0$$

2)

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + e_t$$

3)



D)

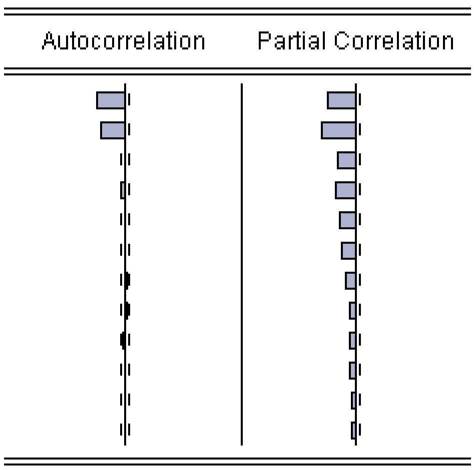
1)

$$SARMA(0,2) \times (0,0)_0$$

2)

$$y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}$$

3)



E)

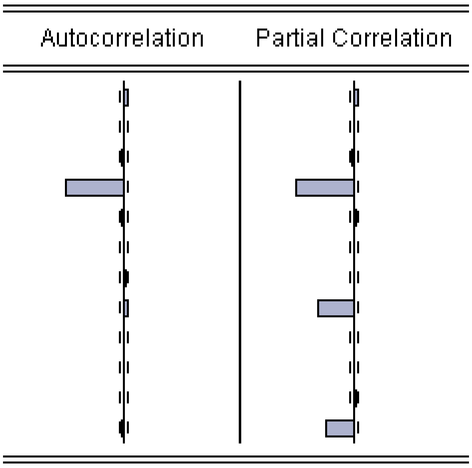
1)

$$SARMA(0,0) \times (0,1)_4$$

2)

$$y_t = e_t - \Theta_1 e_{t-4}$$

3)



F)

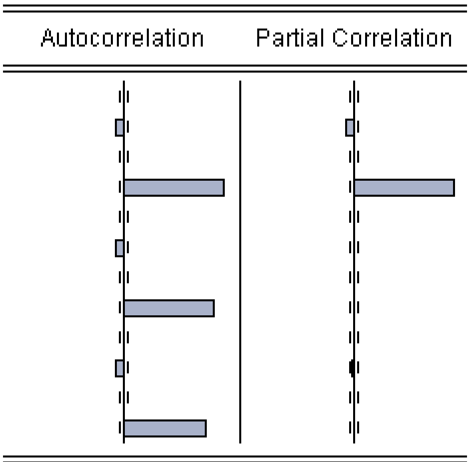
1)

$$SARMA(0,0) \times (1,0)_4$$

2)

$$y_t = \Phi_1 y_{t-4} + e_t$$

3)



Task 3

A)

The lag polynomial is given as $\phi(B) = 1 - \phi B$, so the characteristic equation is

$$1 - \phi z = 0,$$

with solution $z = \frac{1}{\phi}$. The process is covariance stationary if the root is outside the unit circle, that is if $|\frac{1}{\phi}| > 1$.

B)

$$Y_t = \phi Y_{t-1} + e_t$$
$$e_t \sim N(0, \sigma^2)$$

$$E[Y_t] = E[\phi Y_{t-1} + e_t] = \phi E[Y_{t-1}] + E[e_t] = \phi E[Y_{t-1}]$$

assume stationarity, $E[Y_t] = E[Y_{t-1}]$, i.e. $|\phi| < 1$

$$E[Y_t](1 - \phi_1) = 0$$
$$E[Y_t] = \frac{0}{1 - \phi} = 0.$$

C)

$$\gamma_0 = V(Y_t) = V(\phi Y_{t-1} + e_t)$$
$$= \phi^2 V(Y_{t-1}) + V(e_t) + 2\phi Cov(Y_{t-1}, e_t)$$
$$= \phi^2 V(Y_{t-1}) + \sigma^2 + 0,$$

assume stationarity, $V(Y_t) = \gamma_0 \ \forall \ t$, i.e. $|\phi| < 1$

$$\gamma_0 = \phi^2 \gamma_0 + \sigma^2$$
$$\gamma_0 - \phi^2 \gamma_0 = \sigma^2$$
$$\gamma_0(1 - \phi^2) = \sigma^2$$
$$\gamma_0 = \frac{\sigma^2}{1 - \phi^2}.$$

D)

$$\gamma_1 = Cov(Y_t, Y_{t-1})$$

$$= Cov(\phi Y_{t-1} + e_t, Y_{t-1}) = \phi Cov(Y_{t-1}, Y_{t-1}) + Cov(e_t, Y_{t-1})$$

assume stationarity, $V(Y_t) = \gamma_0 \ \forall \ t$, i.e. $|\phi| < 1$

$$\gamma_1 = \phi \gamma_0$$

$$\gamma_2 = Cov(Y_t, Y_{t-2})$$

$$= Cov(\phi Y_{t-1} + e_t, Y_{t-2}) = \phi Cov(Y_{t-1}, Y_{t-2}) + Cov(e_t, Y_{t-2})$$

assume stationarity, $Cov(Y_t, Y_{t-j}) = \gamma_j \ \forall \ t, j$, i.e. $|\phi| < 1$

$$\gamma_2 = \phi \gamma_1$$

$$\gamma_2 = \phi^2 \gamma_0$$

Task 4

A)

We can write the process on lag polynomial form as

$$Y_t = \theta(B) e_t$$

where $\theta(B) = 1 - \theta B$. The characteristic equation is then

$$1 - \theta z = 0$$

with solution $z = \frac{1}{\theta}$. The process is stationary for any finite value of θ. It is invertible if the root is outside the unit circle, that is if $|\frac{1}{\theta}| > 1$.

B)

$$E[Y_t] = E[e_t - \theta e_{t-1}] = E[e_t] - E[\theta e_{t-1}] = E[e_t] - \theta E[e_{t-1}]$$

Using the assumption that error terms are identically distributed and e_t has zero mean we get that

$$E[Y_t] = 0 - \theta \cdot 0 = 0$$

C)

$$\gamma_0 = V(Y_t) = V(e_t - \theta e_{t-1}) = V(e_t) + V(\theta e_{t-1}) - 2\theta Cov(e_t, e_{t-1})$$

Using the assumption that error terms are independent i.e. $Cov(e_t, e_{t-1}) = 0$,

$$V(Y_t) = V(e_t) + V(\theta e_{t-1}) = V(e_t) + \theta^2 V(e_{t-1})$$

Using the assumption that error terms are identically distributed with variance $V(e_t) = V(e_{t-1}) = \sigma^2$ for all t we get

$$\gamma_0 = V(Y_t) = \sigma^2 + \theta^2 \sigma^2 = \sigma^2(1 + \theta^2)$$

D)

For the first autocovariance

$$\gamma_1 = Cov(Y_t, Y_{t-1}) = Cov(e_t - \theta e_{t-1}, e_{t-1} - \theta e_{t-2})$$

$$= Cov(e_t, e_{t-1}) - Cov(e_t, \theta e_{t-2}) - Cov(\theta e_{t-1}, e_{t-1}) + Cov(\theta e_{t-1}, \theta e_{t-2})$$

Taking constants outside the covariances,

$$= Cov(e_t, e_{t-1}) - \theta Cov(e_t, e_{t-2}) - \theta Cov(e_{t-1}, e_{t-1}) + \theta^2 Cov(\theta e_{t-1}, e_{t-2}).$$

Using the assumption that error terms are independent we get $Cov(e_i, e_j) = 0$ for $i \neq j$ and $Cov(e_i, e_j) = \sigma^2$ for $i = j$, we now get that the first autocovariance is

$$\gamma_1 = 0 - \theta \cdot 0 - \theta\sigma^2 + \theta^2 \cdot 0 = -\theta\sigma^2.$$

For the second autocovariance,

$$\gamma_2 = Cov(Y_t, Y_{t-2}) = Cov(e_t - \theta e_{t-1}, e_{t-2} - \theta e_{t-3})$$

$$= Cov(e_t, e_{t-2}) - \theta Cov(e_t, e_{t-3}) - \theta Cov(e_{t-1}, e_{t-2}) + \theta^2 Cov(\theta e_{t-1}, e_{t-3}).$$

Using the assumption that error terms are independent we get

$$\gamma_2 = 0.$$

Task 5

A)

1. Hypothesis

$$H_0 : \{Y_t\} \text{ has a unit root } \Leftrightarrow a = 0$$
$$H_1 : \{Y_t\} \text{ does not have a unit root } \Leftrightarrow a < 0$$

2. Significance level

$\alpha = 0.05$

3. Estimator(s)/Statistics

OLS estimator of a

5. Test statistic

$ADF_{obs} = \dfrac{\hat{a} - 0}{\hat{\sigma}_{\hat{a}}}$

6. Rejection rule and figure

The test statistic doesn't follow a standard distribution so we cannot draw a figure. Reject if p-value < 0.05

4. Assumptions

T is large

7. Calculations and results

P-value from Figure 5.2: $p = 0.8894 > 0.05 \Rightarrow H_0$ is not rejected.

8. Conclusion

We cannot reject the null hypothesis that this process has a unit root at the 5% significance level.

B)

Test of the second autocorrelation

1. Hypothesis

$H_0: \ \ \rho_2 = 0 \quad H_1: \ \ \rho_2 \neq 0$

2. Significance level

$\alpha = 0.05$

3. Estimator

$\hat{\rho}_2$, *the sample autocorrelation for lag 2.*

4. Assumptions

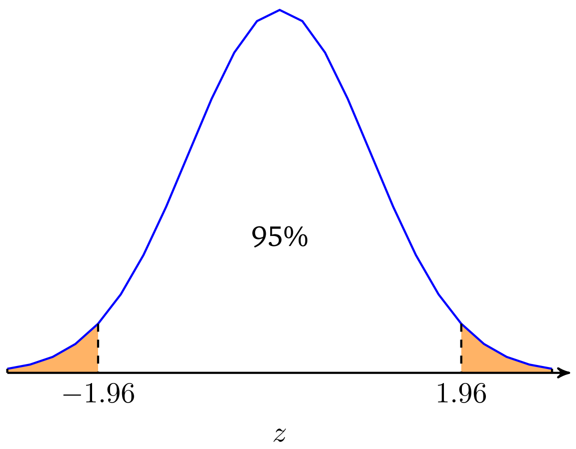
Large T

5. Test statistic

$z_{obs} = \dfrac{\hat{\rho}_2}{\sqrt{1/T}}$, under the null and large $T : z_{obs} \sim N(0,1)$

6. Rejection rule and figure

Reject if: $z_{obs} < -1.96$ or $z_{obs} > 1.96$



7. Calculations and results

$$z_{obs} = \frac{0.159}{\sqrt{1/110}} = 1.668$$

$$-1.96 < z_{obs} < 1.96 \Rightarrow \ H_0 \text{ not rejected}$$

8. Conclusion

We cannot reject the null hypothesis that the second autocorrelation is equal to zero at the 5 % significance level.

C)

1. Hypothesis

$H_0: \quad \rho_1 = \rho_2 = ... = \rho_1 0 = 0$
$H_1: \quad$ At least one $\rho_j \neq 0, j = 1, 2, ..., 10$

2. Significance level

$\alpha = 0.05$

3. Estimator(s)/Statistics

$\hat{\rho}_j, \ \ j = 1, 2, ..., 10$

4. Assumptions

T is large

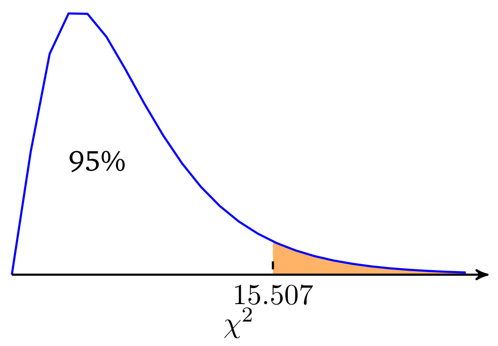
5. Test statistic

$Q_{LB} = T(T+2)\sum_{j=1}^{K} \frac{\hat{\rho}_j^2}{T-j} \sim \chi^2_{K-p-q-P-Q}$

p :	no. of AR-terms
q :	no. of MA-terms
P :	no. of Seasonal AR-terms
Q :	no. of Seasonal MA-terms

6. Rejection rule and figure

Reject if: $Q_{LB} > \chi^2_{8,\,0.05} = 15.507$



7. Calculations and results

From Figure 5: $Q_{LB} = 1.870 < \chi^2_{crit} = 15.507$

8. Conclusion

We cannot reject the null hypothesis that the ten first autocorrelations are simultaneously zero at the 5% significance level.