Statistics B3: Time Series Analysis
Solutions for Exam 2015-04-29

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Task 1

A)

A stochastic process is a sequence of random variables.

B)

In order to be *covariance stationary*, a stochastic process need to fulfill the following

- $E[Y_t] = \mu, \forall\, t$, i.e. the mean is constant over time.
- $V(Y_t) = \gamma_0 < \infty\, \forall\, t$, i.e. the variance is finite and constant over time.
- $Cov(Y_{t+j}, Y_t) = Cov(Y_j, Y_0)\, \forall\, t, j$, i.e. the covariance is independent of *t* and only a function of the lag length *j*.

The third condition actually implies the second condition.

C)

A *trend-stationary* process is (covariance) stationary when removing the trend. The trend should be a function of time only. We can specify such a process by

$$Y_t = \alpha_0 + \alpha_1 t + e_t,$$

where $e_t \sim iid\,(0, \sigma^2)$. By removing the trend we get

$$Y_t - (\alpha_0 + \alpha_1 t) = e_t$$

which is stationary.

D)

A *difference-stationary* process is a process that is (covariance) stationary when taking difference. An example of this is a standard random walk, i.e.

$$Y_t = Y_{t-1} + e_t,$$

where $e_t \sim iid\,(0, \sigma^2)$. Taking first difference we get

$$Y_t - Y_{t-1} = e_t,$$

which is stationary.

E)

- **Identification**: Identify the DGP, or find a DGP that is suitable for modelling the data, this can be done by looking at correlograms.
- **Estimation**: Estimate the parameters of the model, this can be done by OLS or maximum likelihood estimation.
- **Diagnostics**: Investigate to what extent the model captures the systematic variation in the data. The residuals should be white noise, i.e. uncorrelated. This can be done by looking at correlograms and performing statistical tests on the estimated parameters and residuals.
- **Forecasting**: Use the model to forecast conditional mean of the process. Forecasts can be done based on conditional means.

Task 2

$$Y_t = \phi Y_{t-1} + e_t$$
$$e_t \sim N(0, \sigma^2)$$

A)

$$E[Y_t] = E[\phi Y_{t-1} + e_t] = \phi E[Y_{t-1}] + E[e_t] = \phi E[Y_{t-1}]$$

assume stationarity, $E[Y_t] = E[Y_{t-1}]$, i.e. $|\phi| < 1$

$$E[Y_t](1 - \phi_1) = 0$$
$$E[Y_t] = \frac{0}{1 - \phi} = 0$$

B)

$$\gamma_0 = V(Y_t) = V(\phi Y_{t-1} + e_t)$$
$$= \phi^2 V(Y_{t-1}) + V(e_t) + 2\phi Cov(Y_{t-1}, e_t)$$
$$= \phi^2 V(Y_{t-1}) + \sigma^2 + 0,$$

assume stationarity, $V(Y_t) = \gamma_0 \ \forall \ t$, i.e. $|\phi| < 1$

$$\gamma_0 = \phi^2 \gamma_0 + \sigma^2$$

$$\gamma_0 - \phi^2 \gamma_0 = \sigma^2$$

$$\gamma_0 (1 - \phi^2) = \sigma^2$$

$$\gamma_0 = \frac{\sigma^2}{1 - \phi^2}.$$

C)

$$\gamma_1 = Cov(Y_t, Y_{t-1})$$

$$= Cov(\phi Y_{t-1} + e_t, Y_{t-1}) = \phi Cov(Y_{t-1}, Y_{t-1}) + Cov(e_t, Y_{t-1})$$

assume stationarity, $V(Y_t) = \gamma_0 \ \forall \ t$, i.e. $|\phi| < 1$

$$\gamma_1 = \phi \gamma_0$$

$$\gamma_2 = Cov(Y_t, Y_{t-2})$$
$$= Cov(\phi Y_{t-1} + e_t, Y_{t-2}) = \phi Cov(Y_{t-1}, Y_{t-2}) + Cov(e_t, Y_{t-2})$$

assume stationarity, $Cov(Y_t, Y_{t-j}) = \gamma_j \ \forall \ t, j$, i.e. $|\phi| < 1$

$$\gamma_2 = \phi \gamma_1$$

$$\gamma_2 = \phi^2 \gamma_0$$

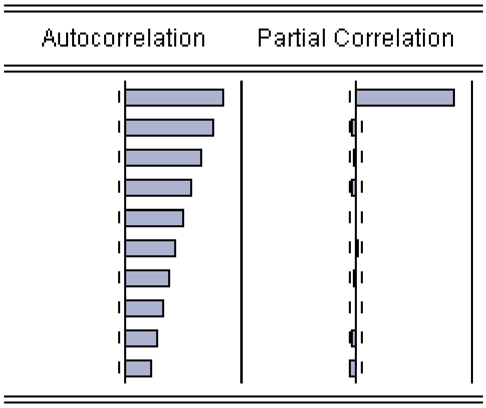
D)

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\phi \gamma_0}{\gamma_0} = \phi$$

$$\rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{\phi^2 \gamma_0}{\gamma_0} = \phi^2$$

E)

ACF and PACF of an AR(1) with $\phi = 0.9$:



Task 3

A)

$$Y_t = (1 - \theta_1 B^1 - \theta_2 B^2)e_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}$$

The expected value is then

$$E[Y_t] = E[e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}] = E[e_t] - \theta_1 E[e_{t-1}] - \theta_2 E[e_{t-2}],$$

Using that $e_t \sim NIID(0, \sigma^2)$ we have

$$E[Y_t] = 0 - \theta_1 \cdot 0 - \theta_2 \cdot 0 = 0.$$

B)

$$V(Y_t) = V(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2})$$

$$= V(e_t) + V(\theta_1 e_{t-1}) - V(\theta_2 e_{t-2}) - 2\theta_1 Cov(e_t, e_{t-1}) - 2\theta_2 Cov(e_t, e_{t-2}) - 2\theta_1\theta_2 Cov(e_{t-1}, e_{t-2})$$

Using that $e_t \sim NIID(0, \sigma^2)$ we have

$$= \sigma^2 + \theta_1^2\sigma^2 + \theta_2^2\sigma^2.$$

$$\gamma_0 = \sigma^2(1 + \theta_1^2 + \theta_2^2).$$

C)

$$\gamma_1 = Cov(Y_t, Y_{t-1}) = Cov(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}, e_{t-1} - \theta_1 e_{t-2} - \theta_2 e_{t-3})$$

$$= Cov(e_t, e_{t-1}) - \theta_1 Cov(e_t, e_{t-2}) - \theta_2 Cov(e_t, e_{t-3}) - \theta_1 Cov(e_{t-1}, e_{t-1})$$

$$+\theta_1^2 Cov(e_{t-1}, e_{t-2}) + \theta_1\theta_2 Cov(e_{t-1}, e_{t-3}) - \theta_2 Cov(e_{t-2}, e_{t-1}) + \theta_2\theta_1 Cov(e_{t-2}, e_{t-2}) + \theta_2^2 Cov(e_{t-2}, e_{t-3})$$

Using that $e_t \sim NIID(0, \sigma^2)$

$$\gamma_1 = -\theta Cov(e_{t-1}, e_{t-1}) + \theta_2\theta_1 Cov(e_{t-2}, e_{t-2})$$

$$= -\theta_1 V(e_{t-1}) + \theta_2\theta_1 V(e_{t-2}) = -\theta_1\sigma^2 + \theta_2\theta_1\sigma^2.$$

$$\gamma_1 = \sigma^2(\theta_2\theta_1 - \theta_1)$$

$$\gamma_2 = Cov(Y_t, Y_{t-2})$$
$$= Cov(e_t, e_{t-2}) - \theta_1 Cov(e_t, e_{t-3}) - \theta_2 Cov(e_t, e_{t-4}) - \theta_1 Cov(e_{t-1}, e_{t-2})$$
$$+\theta_1^2 Cov(e_{t-1}, e_{t-3}) + \theta_1\theta_2 Cov(e_{t-1}, e_{t-4}) - \theta_2 Cov(e_{t-2}, e_{t-2}) + \theta_2\theta_1 Cov(e_{t-2}, e_{t-3}) + \theta_2^2 Cov(e_{t-2}, e_{t-4})$$

Using that $e_t \sim NIID(0, \sigma^2)$

$$\gamma_2 = -\theta_2 V(e_{t-2}) = -\theta_2\sigma^2.$$

$$\gamma_3 = Cov(Y_t, Y_{t-3})$$
$$= Cov(e_t, e_{t-3}) - \theta_1 Cov(e_t, e_{t-4}) - \theta_2 Cov(e_t, e_{t-5}) - \theta_1 Cov(e_{t-1}, e_{t-3})$$
$$+\theta_1^2 Cov(e_{t-1}, e_{t-4}) + \theta_1\theta_2 Cov(e_{t-1}, e_{t-5}) - \theta_2 Cov(e_{t-2}, e_{t-3}) + \theta_2\theta_1 Cov(e_{t-2}, e_{t-4}) + \theta_2^2 Cov(e_{t-2}, e_{t-5})$$

Using that $e_t \sim NIID(0, \sigma^2)$

$$\gamma_3 = 0.$$

D)

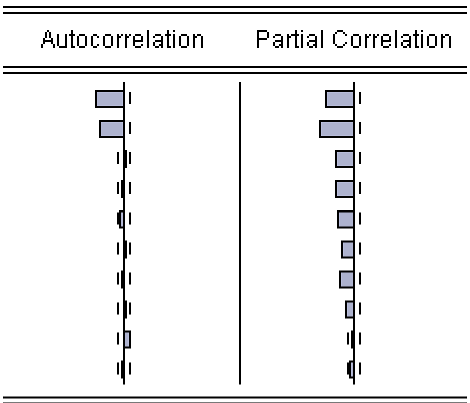
$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\sigma^2(\theta_2\theta_1 - \theta_1)}{\sigma^2(1 + \theta_1^2 + \theta_2^2)} = \frac{\theta_2\theta_1 - \theta_1}{1 + \theta_1^2 + \theta_2^2}.$$

$$\rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{-\theta_2\sigma^2}{\sigma^2(1 + \theta_1^2 + \theta_2^2)} = -\frac{\theta_2}{1 + \theta_1^2 + \theta_2^2}.$$

$$\rho_3 = \frac{\gamma_3}{\gamma_0} = \frac{0}{\sigma^2(1 + \theta_1^2 + \theta_2^2)} = 0.$$

E)

ACF and PACF of a MA(2) with $\theta_1 = 0.5$ and $\theta_2 = 0.3$:



F)

The process is invertible for $|\theta| < 1$.

With recursion,

$$\begin{aligned} Y_t &= e_t - \theta \quad e_{t-1} \\ &= e_t - \theta(Y_{t-1} + \theta e_{t-2}) \\ &= e_t - \theta(Y_{t-1} + \theta(Y_{t-2} + \theta e_{t-3})) \\ &= e_t - \theta(Y_{t-1} + \theta(Y_{t-2} + \theta(Y_{t-3} + \theta e_{t-4}))) \\ &\vdots \end{aligned}$$

$$Y_t = e_t - \sum_{i=1}^{\infty} \theta^i Y_{t-1}$$

$$\Rightarrow Y_t = e_t - \theta Y_{t-1} - \theta^2 Y_{t-2} - \theta^3 Y_{t-3} - \theta^4 Y_{t-4} - \theta^5 Y_{t-5} - ...$$

We can also use the concept of geometric series and write

$$Y_t = (1 - \theta B)e_t$$
$$e_t = \frac{Y_t}{(1 - \theta B)}$$

which is the sum of the infinite series

$$e_t = \sum_{i=0}^{\infty} (\theta B)^i Y_{t-i} = Y_t + \sum_{i=1}^{\infty} (\theta B)^i Y_{t-i}.$$

By rearranging

$$Y_t = e_t - \sum_{i=1}^{\infty} (\theta B)^i Y_{t-i}$$

$$Y_t = e_t - \theta Y_{t-1} - \theta^2 Y_{t-2} - \theta^3 Y_{t-3} - \theta^4 Y_{t-4} - \theta^5 Y_{t-5} - ...$$

Task 4

A)

The estimate of the correlation between Y_t and Y_{t-2} is 0.977

B)

The estimate of the correlation between Y_t and Y_{t-2} when taking the correlation between Y_t and Y_{t-1} into account is -0.013.

C)

1. Hypotheses

$$H_0 : \{Y_t\} \text{ has a unit root } \quad \Leftrightarrow a = 0$$
$$H_1 : \{Y_t\} \text{ does not have a unit root } \quad \Leftrightarrow a < 0$$

2. Significance level

$\alpha = 0.05$

3. Estimator(s)/Statistics

OLS estimator of a

4. Assumptions

T is large

5. Test statistic

$$ADF_{obs} = \frac{\hat{a} - 0}{\hat{\sigma}_{\hat{a}}}$$

6. Rejection rule and figure

The test statistic doesn't follow a standard distribution so we cannot draw a figure. Reject if p-value < 0.05

7. Calculations and results

P-value from Figure 5.3: $p = 0.5533 > 0.05 \Rightarrow H_0$ is not rejected.

8. Conclusion

We cannot reject the null hypothesis that the raw data contains a unit root at the 5% significance level.

D)

1. Hypotheses

$$H_0: \quad \rho_1 = \rho_2 = \rho_3 = 0$$
$$H_1: \quad \text{At least one } \rho_j \neq 0, j = 1, 2, 3$$

2. Significance level

$\alpha = 0.01$

3. Estimator(s)/Statistics

$\hat{\rho}_j, \quad j = 1, 2, 3$

4. Assumptions

T is large

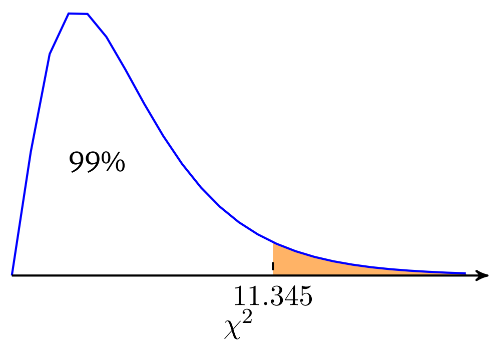
5. Test statistic

$$Q_{LB} = T(T+2)\sum_{j=1}^{K} \frac{\hat{\rho}_j^2}{T-j} \sim \chi^2_{K-p-q-P-Q}$$

p : no. of AR-terms
q : no. of MA-terms
P : no. of Seasonal AR-terms
Q : no. of Seasonal MA-terms

6. Rejection rule and figure

Reject if: $\quad Q_{LB} > \chi^2_{3,\,0.01} = 11.345$



7. Calculations and results

From Figure 5.9: $Q_{LB} = 57.006 > \chi^2_{crit} = 11.345$

8. Conclusion

We reject the null hypothesis that the first three autocorrelations are simultaneously zero at the 1% significance level.

E)

1. Hypotheses

$H_0 : \rho_1 = 0$
$H_1 : \rho_1 > 0$

2. Significance level

$\alpha = 0.01$

3. Estimator(s)/Statistics

$\hat{\rho}_1$

4. Assumptions

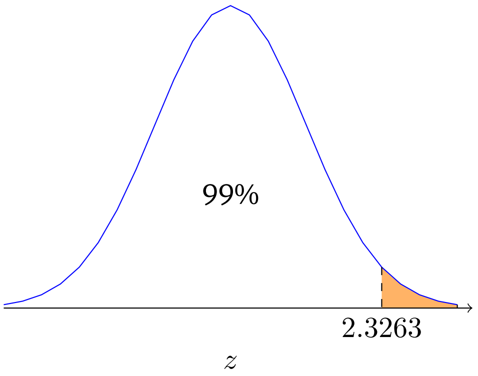
T is large

5. Test statistic

$z_{obs} = \frac{\hat{\rho}_1 - \rho_1^{H_0}}{\sigma_{\hat{\rho}_1}} \sim N(0,1)$ where $\sigma_{\hat{\rho}_1} = \sqrt{\frac{1}{T}}$

6. Rejection rule and figure

Reject if: $z_{obs} > 2.3263$



7. Calculations and results

$z_{obs} = \frac{\hat{\rho}_1 - \rho_1^{H_0}}{\sigma_{\hat{\rho}_1}} = \frac{-0.072-0}{\sqrt{1/238}} = -1.111$

$z_{obs} < z_{crit} = 2.3263 \Rightarrow H_0$ is not rejected.

8. Conclusion

We cannot reject the null hypothesis that the first autocorrelation is zero at the 1% significance level.

F)

When testing only the first autocorrelation we cannot reject the null (that it is zero). However, the first three autocorrelations are significantly different from zero based on the Q-statistic in Figure 4.7 (p-value equal to 0.002). We cannot claim that we have extracted all the systematic variation in the residuals.

G)

When examining the residuals from the ARMA(1,1)-model, none of the estimated autocorrelations are significantly different from zero so this model captures more systematic variation. Both parameter estimates are significant and the adjusted R-squared is higher. The ARMA(1,1)-model is better.

Task 5

A)

$$
\begin{aligned}
D &= 0 \\
d &= 0 \\
s &= 0 \\
\phi(B) &= (1 - \phi_1 B - \phi_2 B^2) \\
\theta(B) &= (1 - \theta B) \\
\Theta(B) &= 1
\end{aligned}
$$

B)

$$(1 - \phi_1 B - \phi_2 B^2) Y_t = (1 - \theta B) e_t$$
$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} - \theta e_{t-1} + e_t$$

C)

$$
\begin{aligned}
D &= 0 \\
d &= 0 \\
s &= 4 \\
\phi(B) &= 1 \\
\theta(B) &= (1 - \theta B) \\
\Theta(B) &= (1 - \Theta B^4)
\end{aligned}
$$

D)

$$Y_t = (1 - \Theta B^4)(1 - \theta B) e_t$$
$$Y_t = (1 - \theta B - \Theta B^4 + \Theta \theta B^5) e_t$$
$$Y_t = e_t - \theta e_{t-1} - \Theta e_{t-4} + \Theta \theta e_{t-5}$$