EXAM Probability theory and statistical inference I , 2ST065 (7.5 hp).
Wednesday 28/9 2016, 8.00 - 13.00.
Examiner: Patrik Andersson.

Allowed tools.
– Formulae for the course Probability Theory and Statistical Inference
– Math Handout (by Lars Forsberg)
– Pocket calculator.
– Physical dictionary (or word-list).

Notes in the permitted aids are not allowed. If you feel that something in the wording of the problem is unclear, write under what assumptions you are solving the problem. After turning in your test, you may keep the test-pages with the question-statements. For the grade Pass, a score of at least 50% is required on the exam.

1. A sales person is selling magazine subscriptions by telephone. In 20% of the phone calls, a subscription gets sold.
   (a) Assuming that the person makes three phone calls per day, what is the probability that in a five day working week, more than 3 subscriptions get sold? (7p)
   (b) What is the expected number of magazine subscriptions sold in a 47 week work year? (7p)
   (c) The sales person gets 20 kronor for every subscription sold. What is the standard deviation of the yearly salary? (6p)
2. Let Y_1 and Y_2 have the joint density function
$$p_{Y_1,Y_2}(y_1, y_2) = Cy_1(y_1 + y_2),\ 0 < y_1 < 1, 0 < y_2 < 2,$$
   and 0 elsewhere.
   (a) Find C such that this is a true density. (6p)
   (b) Find $E[Y_1|Y_2 = y_2]$. (7p)
   (c) Find $P(Y_1 \leq 0.5|Y_2 = 1)$. (7p)
3. Let $X \sim \mathsf{Exp}(\beta)$. The random variable Y is said to have a shifted exponential distribution if $Y = X + L$, for some constant L.
   (a) Find the density of Y. (10p)

(b) Assume that $y_1, y_2, \ldots, y_n$ is a sample from Y. Find the method of moments estimators of β and L. (10p)

4. After sampling from a $\mathsf{N}(\mu, \sigma^2)$ distribution the following measurements were obtained.

Observation #	1	2	3	4	5	6	7	8	9	10
Measurement	-3.74	9.40	2.47	0.11	2.42	-0.31	-0.07	4.76	4.52	4.55

(a) Test on the 5%-level if $\mu = 0$, against $\mu \neq 0$. (7p)

(b) Find the approximate p-value of the test. (7p)

(c) Test on the 10%-level if $\sigma^2 = 1$, against $\sigma^2 \neq 1$. (6p)

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1. (a) In a work week 15 phone calls are made. Let X be the number of sold subscriptions, so that $X \sim \mathsf{Bin}(15, 0.2)$. Thus,

$$P(X > 3) = 1 - P(X \leq 3) = 1 - 0.6482 = 0.3518,$$

where we used Table 4.1.

(b) In a work year 705 phone calls are made. If now X is the number of sold subscriptions in a year, $X \sim \mathsf{Bin}(705, 0.2)$ and thus $E[X] = 705 \cdot 0.2 = 141$.

(c) The yearly salary will be $20X$ so that

$$\mathrm{Var}(20X) = 400\mathrm{Var}(X) = 400 \cdot 705 \cdot 0.2(1 - 0.2) = 45120.$$

The standard deviation is then $\sqrt{45120} \approx 212$.

2. (a) Since,

$$\int_0^1 \int_0^2 y_1(y_1 + y_2)dy_2dy_1 = \int_0^1 y_1 \left[y_1y_2 + \frac{y_2^2}{2}\right]_0^2 dy_1 = \int_0^1 y_1(2y_1 + 2)dy_1$$
$$= \left[2\frac{y_1^3}{3} + 2\frac{y_1^2}{2}\right]_0^1 = \frac{5}{3},$$

we need to have $c = \frac{3}{5}$.

(b) We start by finding the conditional density,

$$p_{Y_1|Y_2}(y_1, y_2) = \frac{p_{Y_1,Y_2}(y_1, y_2)}{p_{Y_2}(y_2)} = cy_1(y_1 + y_2),\ 0 < y_1 < 1.$$

We find c similarly to (a), that is by calculating

$$\int_0^1 y_1(y_1 + y_2)dy_1 = \left[\frac{y_1^3}{3} + y_2\frac{y_1^2}{2}\right]_0^1 = \frac{2 + 3y_2}{6}.$$

So that

$$p_{Y_1|Y_2}(y_1, y_2) = \frac{6}{2 + 3y_2}y_1(y_1 + y_2),\ 0 < y_1 < 1.$$

Finally we get

$$E\left[Y_1|Y_2 = y_2\right] = \int_0^1 y_1\frac{6}{2 + 3y_2}y_1(y_1 + y_2)dy_1 = \frac{6}{2 + 3y_2}\left[\frac{y_1^4}{4} + y_2\frac{y_1^3}{3}\right]_0^1$$
$$= \frac{3 + 4y_2}{4 + 6y_2}.$$

(c) Having already done the calculation in (b) we immediately get that

$$p_{Y_1|Y_2}(y_1, 1) = \frac{6}{5} y_1(y_1 + 1),\ 0 < y_1 < 1.$$

Therefore,

$$P(Y \leq 0.5|Y_2 = 1) = \int_0^{0.5} p_{Y_1|Y_2}(y_1, 1)dy_1 = \int_0^{0.5} \frac{6}{5} y_1(y_1 + 1)dy_1$$
$$= \frac{6}{5}\left[\frac{y_1^3}{3} + \frac{y_1^2}{2}\right]_0^{0.5} = \frac{1}{5}.$$

3. (a) From the table we get that $f_X(x) = e^{-x/\beta}/\beta,\ x > 0$. Then,

$$F_Y(y) = P(Y \leq y) = P(X + L \leq y) = P(X \leq y - L) = F_X(y - L),$$

so that

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}F_X(y - L) = \frac{F_X(y - L)}{d(y - L)}\frac{d(y - L)}{dy} = f_Y(y - L)$$
$$= \frac{e^{-(y-L)/\beta}}{\beta},\ y > L.$$

(b) The first two moments are

$$\mu_1 = E[Y] = E[X + L] = \beta + L, \tag{1}$$
$$\mu_2 = E\left[Y^2\right] = Var(Y) + E[Y]^2 = Var(X + L) + (\beta + L)^2$$
$$= Var(X) + (\beta + L)^2 = \beta^2 + (\beta + L)^2. \tag{2}$$

Equation (1) gives $L = \mu_1 - \beta$ which inserted in (2) gives

$$\mu_2 = \beta^2 + \mu_1^2 \implies \beta = \sqrt{\mu_2 - \mu_1^2},$$

where we chose the positive root since $\beta > 0$ in the exponential distribution. We thus have the method of moments estimators

$$\hat{L} = \hat{\mu}_1 - \sqrt{\hat{\mu}_2 - \hat{\mu}_1^2} = \bar{y} - \sqrt{\frac{1}{n}\sum_i y_i^2 - \bar{y}^2},$$
$$\bar{\beta} = \sqrt{\frac{1}{n}\sum_i y_i^2 - \bar{y}^2}.$$

4. (a) We first find the mean to be 2.41 and sample standard deviation to be 3.65. The test statistic is $\frac{2.41-0}{3.65/\sqrt{10}} \approx 2.09$. This is t-distributed with 9 d.o.f. under H_0. We are doing a double-sided test on the 5%-level, so we should compare to $t_{0.025}(9) = 2.262$. Since the observed statisic is smaller then the critical value, we do not reject H_0.

   (b) We observe that

$$t_{0.05}(9) = 1.833 < 2.09 < 2.262 = t_{0.025}(9),$$

   and thus the p-value is between 5% and 10%.

   (c) Here the test-statistic is

$$\frac{(n-1)s^2}{\sigma_0^2} = \frac{9 \cdot (3.65)^2}{1} \approx 119.$$

   If $\sigma^2 = 1$, this is $\chi^2(9)$-distributed. Since the test is double sided and the level is 10%, the statistic should be compared to $\chi^2_{0.05}(9) = 16.9$ and $\chi^2_{0.95}(9) = 3.3$. Since $\chi^2_{0.05}(9) = 16.9 < 39$ we reject $\sigma^2 = 1$ in favor of $\sigma^2 \neq 1$.