Written Examination in Time Series Analysis
2017-04-28
Solutions

Task 1

A.

A stochastic process is a sequence of random variables.

B.

In order to be covariance stationary, a stochastic process needs to fulfill the following:

1. $E(Y_t) = \mu$, $\forall t$. The mean is constant over time.
2. $Var(Y_t) = \gamma_0 < \infty$, $\forall t$. The variance is finite and constant over time.
3. $Cov(Y_t, Y_{t+k}) = Cov(Y_0, Y_k)$, $\forall t, k$. The covariance is independent of t and is only a function of the lag length, k.

The third condition actually implies the second.

C.

The process of interest is

$$Y_t = \phi Y_{t-1} + \alpha_0 + \alpha_1 t + e_t - \theta e_{t-1}$$

where $e_t \sim NID(0, \sigma^2)$.

The process has a stochastic trend for all α_0, $\alpha_1 = 0$, $|\phi| \geq 1$ and for all θ. The process is difference-stationary i.e. the first difference is a covariance stationary process.

D.

The process has a deteministic trend for every α_0, $\alpha_1 \neq 0$, $|\phi| < 1$ and for all θ. The first difference is a covariance stationary process, or by removing the trend we get $Y_t - \alpha_1 t = \phi Y_{t-1} + \alpha_0 + e_t - \theta e_{t-1}$, which is a stationary process.

E.

1. Identication: Identify the DGP, or find a DGP that is suitable for modelling the data, this can be done by looking at correlograms.

2. Estimation: Estimate the parameters of the model, this can be done by OLS or maximum likelihood estimation.

3. Diagnostics: Investigate to what extent the model captures the systematic variation in the data. The residuals should be white noise, i.e. uncorrelated. This can be done by looking at correlograms and plots and performing statistical tests on the estimated parameters and residuals.

4. Forecasting: Use the model to forecast the conditional mean of the process. Forecasts can be done based on conditional means.

Task 2

The process of interest is

$$Y_t = \phi Y_{t-1} + e_t \tag{1}$$

where $e_t \sim NID(0, \sigma^2)$. That is, an AR(1) process.

A.

The characteristic equation for (1) is $1 - \phi x = 0$. Solving for x we get

$$x = \frac{1}{\phi} \tag{2}$$

B.

The AR(1) process is stationary if the root of the characteristic equation is greater than one in absolute values, i.e. when $|x| > 1$. This is true whenever $|\phi| < 1$.

C.

We can rewrite (1) as

$$\begin{aligned} Y_t &= \phi Y_{t-1} + e_t \\ (1 - B\phi) Y_t &= e_t \\ Y_t &= (1 - B\phi)^{-1} e_t \end{aligned} \tag{3}$$

We make use of the assumption that $|\phi| < 1$, which is equivalent of $|B\phi| < 1$. Under this assumption, we use geometric series and the fact that

$$\frac{1}{(1 - B\phi)} = \sum_{j=0}^{\infty} (B\phi)^j \tag{4}$$

so Y_t can be rewritten as

$$\begin{aligned} Y_t &= \sum_{j=0}^{\infty}(B\phi)^j e_t \\ &= \sum_{j=0}^{\infty} \phi^j e_{t-j} \\ &= e_t + \sum_{j=1}^{\infty} \theta_j e_{t-j} \end{aligned} \tag{5}$$

which is a $MA(\infty)$ representation with $\theta_j = \phi^j$.

D.

The process is now

$$Y_t = Y_{t-1} + e_t \tag{6}$$

that is, a random walk. With $Y_0 = 0$, the process can be written as

$$Y_t = \sum_{j=1}^{t} e_t \tag{7}$$

so the expected value is

$$\begin{aligned} E(Y_t) &= E(e_1) + E(e_2) + ... + E(e_t) \\ &= 0 \end{aligned} \tag{8}$$

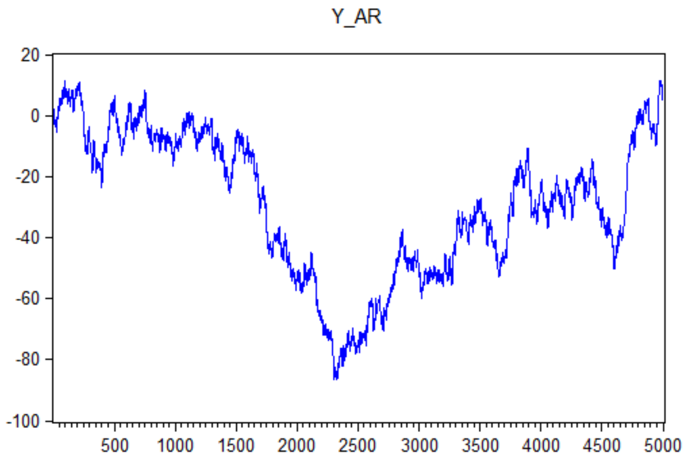
E.

The variance of the process in subtask D is, for a general t,

$$\begin{aligned} Var(Y_t) &= Var(e_1) + Var(e_2) + ... + Var(e_t) + 2Cov(e_1, e_2) + ... + 2Cov(e_{t-1}, e_t) \\ &= \sigma^2 + \sigma^2 + ... + \sigma^2 + 0 + ... + 0 \\ &= \sum_{j=1}^{t} \sigma^2 \\ &= t\sigma^2 \end{aligned} \tag{9}$$

F.

Figure 1: A simulated random walk



G.

Figure 2: SACF and SPACF of a random walk

Date: 05/03/17 Time: 08:45
Sample: 1 10000
Included observations: 10000

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	1.000	1.000	9996.6	0.000
		2	0.999	-0.006	19988.	0.000
		3	0.999	0.006	29973.	0.000
		4	0.999	0.004	39954.	0.000
		5	0.998	-0.014	49928.	0.000
		6	0.998	-0.014	59898.	0.000
		7	0.998	-0.010	69861.	0.000
		8	0.997	0.008	79818.	0.000
		9	0.997	0.003	89770.	0.000
		10	0.997	0.003	99716.	0.000
		11	0.996	-0.002	109657	0.000
		12	0.996	-0.002	119592	0.000
		13	0.996	-0.003	129521	0.000
		14	0.995	-0.002	139444	0.000
		15	0.995	-0.001	149362	0.000

Task 3

The process of interest is

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} \tag{10}$$

where $e_t \sim NID(0, \sigma^2)$. That is, a MA(2) process.

A.

The expected value of the process is

$$\begin{aligned} E(Y_t) &= E(e_t) - \theta_1 E(e_{t-1}) - \theta_2 E(e_{t-2}) \\ &= 0 \end{aligned} \tag{11}$$

B.

The variance, γ_0, of the process is

$$\begin{aligned} \gamma_0 &= Var(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}) \\ &= Var(e_t) + \theta_1^2 Var(e_{t-1}) + \theta_2^2 Var(e_{t-2}) \\ &- 2\theta_1 Cov(e_t, e_{t-1}) - 2\theta_2 Cov(e_t, e_{t-2}) + 2\theta_1\theta_2 Cov(e_{t-1}, e_{t-2}) \\ &\text{since the error terms are assumed to be independent, the cov. terms are all zero and} \\ \gamma_0 &= \sigma^2 + \theta_1^2\sigma^2 + \theta_1^2\sigma^2 \\ &= \sigma^2(1 + \theta_1^2 + \theta_2^2) \end{aligned} \tag{12}$$

C.

The first autocovariance is

$$
\begin{aligned}
\gamma_1 &= Cov(Y_t, Y_{t-1}) \\
&= Cov(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}, e_{t-1} - \theta_1 e_{t-2} - \theta_2 e_{t-3}) \\
&= Cov(e_t, e_{t-1}) - \theta_1 Cov(e_t, e_{t-2}) - \theta_2 Cov(e_t, e_{t-3}) \\
&- \theta_1 Var(e_{t-1}) + \theta_1^2 Cov(e_{t-1}, e_{t-2}) + \theta_1\theta_2 Cov(e_{t-1}, e_{t-3}) \\
&- \theta_2 Cov(e_{t-2}, e_{t-1}) + \theta_1\theta_2 Var(e_{t-2}) + \theta_2^2 Cov(e_{t-2}, e_{t-3})
\end{aligned}
$$

since the error terms are assumed to be independent, the cov. terms are all zero and

$$
\begin{aligned}
\gamma_1 &= \theta_1\theta_2 Var(e_{t-2}) - \theta_1 Var(e_{t-1}) \\
&= \sigma^2(\theta_1\theta_2 - \theta_1)
\end{aligned}
\tag{13}
$$

The second autocovariance is

$$
\begin{aligned}
\gamma_2 &= Cov(Y_t, Y_{t-2}) \\
&= Cov(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}, e_{t-2} - \theta_1 e_{t-3} - \theta_2 e_{t-4}) \\
&= Cov(e_t, e_{t-2}) - \theta_1 Cov(e_t, e_{t-3}) - \theta_2 Cov(e_t, e_{t-4}) \\
&- \theta_1 Cov(e_{t-1}, e_{t-2}) + \theta_1^2 Cov(e_{t-1}, e_{t-3}) + \theta_1\theta_2 Cov(e_{t-1}, e_{t-4}) \\
&- \theta_2 Var(e_{t-2}) + \theta_1\theta_2 Cov(e_{t-2}, e_{t-3}) + \theta_2^2 Cov(e_{t-2}, e_{t-4})
\end{aligned}
$$

since the error terms are assumed to be independent, the cov. terms are all zero and

$$
\begin{aligned}
\gamma_2 &= -\theta_2 Var(e_{t-2}) \\
&= -\theta_2\sigma^2
\end{aligned}
\tag{14}
$$

D.

See Figure 3.

E.

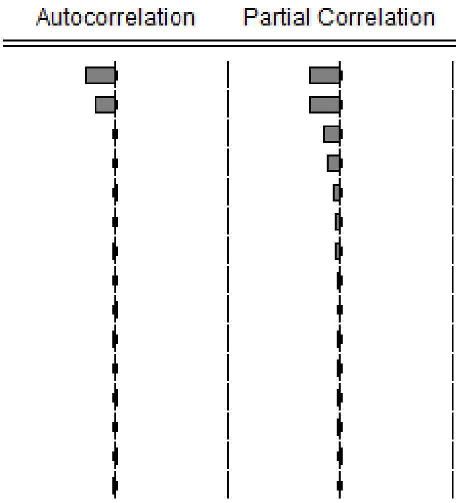
We are to calculate the conditional expected value of the process at time $t+1$, given all the information up to and including time t. That is,

$$
\begin{aligned}
\hat{Y}_{t+1} &= E[Y_{t+1}|I_t] \\
&= E[e_{t+1}|e_t, e_{t-1}, ...] - \theta_1 E[e_t|e_t, e_{t-1}, ...] - \theta_2 E[e_{t-1}|e_t, e_{t-1}, ...] \\
&= -\theta_1 e_t - \theta_2 e_{t-1}
\end{aligned}
\tag{15}
$$

since we can use the fact that $E[e_{t+1}|e_t, e_{t-1}, ...] = 0$ and that the expected value of a constant is the constant itself.

Figure 3: SACF and SPACF of a MA(2) process

Date: 05/03/17 Time: 09:18
Sample: 1 10000
Included observations: 10000



	AC	PAC	Q-Stat	Prob
1	-0.263	-0.263	691.40	0.000
2	-0.178	-0.265	1007.7	0.000
3	-0.002	-0.151	1007.7	0.000
4	0.004	-0.110	1007.8	0.000
5	0.009	-0.064	1008.6	0.000
6	0.005	-0.037	1008.9	0.000
7	-0.017	-0.042	1011.7	0.000
8	0.005	-0.022	1011.9	0.000
9	0.012	-0.006	1013.3	0.000
10	-0.019	-0.025	1016.9	0.000
11	-0.003	-0.019	1017.0	0.000
12	0.009	-0.008	1017.9	0.000
13	-0.005	-0.013	1018.1	0.000
14	0.014	0.008	1019.9	0.000
15	-0.011	-0.007	1021.1	0.000

F.

The expected value of the forecast error is

$$\begin{aligned}E[Y_{t+1} - \hat{Y}_{t+1}] &= E[e_{t+1} - \theta_1 e_t - \theta_2 e_{t-1} - (-\theta_1 e_t - \theta_2 e_{t-1})] \\ &= E[e_{t+1}] \\ &= 0\end{aligned} \tag{16}$$

G.

The forecast error variance is

$$\begin{aligned}Var[Y_{t+1} - \hat{Y}_{t+1}] &= Var[e_{t+1} - \theta_1 e_t - \theta_2 e_{t-1} - (-\theta_1 e_t - \theta_2 e_{t-1})] \\ &= Var[e_{t+1}] \\ &= \sigma^2\end{aligned} \tag{17}$$

Task 4

We consider the real dataset, US GDP 1980 - 2009, quartely data.

A.

We follow the test template for the test of whether the original data contains a unit root or not.

1. Hypotheses

$H_0 : \{Y_t\}$ has a unit root $\Leftrightarrow a = 0$
$H_1 : \{Y_t\}$ does not have a unit root $\Leftrightarrow a < 0$

2. Significance level

$\alpha = 0.05$

3. Estimator(s)/Statistics

OLS estimator of a in $\bigtriangledown Y_t = \phi_1 \bigtriangledown Y_{t-1} + ... + \phi_k \bigtriangledown Y_{t-k} + e_t$

4. Assumptions

T is large.

5. Test statistic

$$ADF_{obs} = \frac{\hat{a} - 0}{\hat{\sigma}_{\hat{a}}}$$

6. Rejection rule and figure

The null hypothesis is rejected if the p-value $< \alpha = 0.05$. We cannot draw a figure since the test statistic does not follow a standard distribution.

7. Calculations and results

From Figure 4.2, we can see that the p-value $= 0.8635 > 0.05$. We cannot reject the null hypothesis.

8. Conclusion

We cannot reject the null hypothesis that the original data contains a unit root at a 5 percent significance level.

B.

A test if the fourth autocorrelation is different from zero.

1. Hypotheses

$H_0 : \rho_4 = 0$
$H_1 : \rho_4 \neq 0$

2. Significance level

$\alpha = 0.05$

3. Estimator(s)/Statistics

$\hat{\rho}_4$, the sample autocorrelation for lag 4.

4. Assumptions

T is large

5. Test statistic

$$z_{obs} = \frac{\hat{\rho}_k - \rho_k^{H_0}}{\sqrt{\frac{1}{T}}} \sim N(0,1)$$

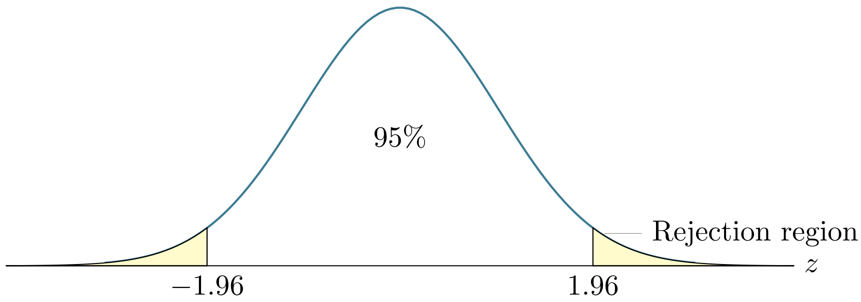
6. Rejection rule and figure

Reject the null hypothesis if $z_{obs} < -1.96$ or if $z_{obs} > 1.96$.

7. Calculations and results

$$z_{obs} = \frac{-0.257 - 0}{\sqrt{\frac{1}{114}}} = -2.74... < 1.96$$

We reject the null hypothesis.



8. Conclusion

We reject the null hypothesis that the fourth autocorrelation is zero at the 5 percent significance level.

C.

A test whether the first three autocorrelations are simultaneously zero.

1. Hypotheses

$H_0 : \rho_1 = \rho_2 = \rho_3 = 0$
$H_1 :$ At least one $\rho_j \neq 0, \; j = 1, 2, 3$

2. Significance level

$\alpha = 0.05$

3. Estimator(s)/Statistics

$\hat{\rho}_j, \; j = 1, 2, 3$

4. Assumptions

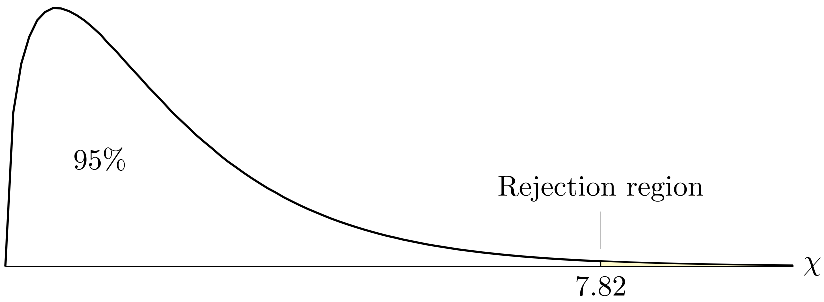
T is large

5. Test statistic

$Q_{LB} = T(T+2) \sum_{j=1}^{K} \dfrac{\hat{\rho}_j^2}{T-j} \sim \chi_{K-p-q-P-Q}$

6. Rejection rule and figure

Reject the null hypothesis if $\chi_{obs} > \chi_{3,0.05} = 7.82$



7. Calculations and results

$$
\begin{aligned}
Q_{LB} &= 114 * 116 * \left(\frac{(-0.053)^2}{114-1} + \frac{(0.141)^2}{114-2} + \frac{(-0.091)^2}{114-3} \right) \\
&= 3.66... \\
&< 7.82
\end{aligned} \tag{18}
$$

We cannot reject the null hypothesis.

8. Conclusion

We cannot reject the null hypothesis that the first three autocorrelations are zero at the 5 percent significance level.

D.

The fourth sample autocorrelation is the correlation between the realization of a process at a given time point and the same process four time points back.

E.

The eight sample partial autocorrelation is the correlation between the the realization of a process at a given time point and the same process eight time points back, when taking the autocorrelations 1-7 into account